October Math Gems

Problem of the week 30

§1 Problems

Problem 1.1. Of 450 students assembled for a concert, 40 percent were boys. After a bus containing an equal number of boys and girls brought more students to the concert, 41 percent of the students at the concert were boys. Find the number of students on the bus.

Solution. Of the 450 students originally at the concert, $(0.40)450 = 180$ were boys. Let n be the number of boys on the bus, so the number of students on the bus was $2n$. Then $0.41 = \frac{180+n}{450+2n}$ which simplifies to $(0.41)450 + 0.82n = 180 + n$ and $n = \frac{0.41\times 450-180}{1-0.82} = \frac{4.5}{0.18} = 25$. Thus, the number of students on the bus was $2 \times 25 = 50$. □

Problem 1.2. Let a be a positive real number such that

$$4a^2 + \frac{1}{a^2} = 117.$$

Find

$$8a^3 + \frac{1}{a^3}.$$

Solution. Adding 4 to each side of the given equation yields

$$4a^2 + 4 + \frac{1}{a^2} = 121 = 11^2,$$

so $2a + \frac{1}{a} = 11$. Hence, cubing both sides yields

$$11^3 = 8a^3 + 3\times 4a + 3\times\frac{2}{a} + \frac{1}{a^3} = 8a^3 + \frac{1}{a^3} + 6(2a + \frac{1}{a}) = 8a3 + \frac{1}{a^3} + 6\times 11.$$

It follows that

$$8a^3 + \frac{1}{a^3} = 11^3 - 6\times 11 = 1331 - 66 = 1265.$$

□

Problem 1.3. Find $x^6 + \frac{1}{x^6}$ if $x + \frac{1}{x} = 3$.

Solution.

$$\therefore (x + \frac{1}{x})^2 = 9$$

$$\therefore x^2 + (\frac{1}{x})^2 = 7$$

$$\therefore (x^2 + (\frac{1}{x})^2)^3 = (7)^3$$

$$\therefore x^6 + (\frac{1}{x})^6 + 3(x^2 + \frac{1}{x^2}) = (7)^3$$
$$\therefore x^6 + (\frac{1}{x})^6 = (7)^3 - 3(7) = 322$$

□

Problem 1.4. Let $a_1 = 2021$ and for $n \geq 1$ let $a_{n+1} = \sqrt{4 + a_n}$. Then a_5 can be written as

$$\sqrt{\frac{m+\sqrt{n}}{2}} + \sqrt{\frac{m-\sqrt{n}}{2}}$$

where m and n are positive integers. Find $10m + n$.

Solution. One can calculate

$$\begin{aligned}
a_2 &= \sqrt{2021+4} = \sqrt{2025} = 45 \\
a_3 &= \sqrt{45+4} = \sqrt{49} = 7 \\
a_4 &= \sqrt{7+4} = \sqrt{11} \\
a_5 &= \sqrt{\sqrt{11}+4}.
\end{aligned}$$

Then m and n must satisfy

$$(\sqrt{\sqrt{11}+4})^2 = \left(\sqrt{\frac{m+\sqrt{n}}{2}} + \sqrt{\frac{m-\sqrt{n}}{2}}\right)^2$$

so

$$4+\sqrt{11} = \frac{m+\sqrt{n}}{2} + 2\left(\sqrt{\frac{m+\sqrt{n}}{2}}\right)\left(\sqrt{\frac{m-\sqrt{n}}{2}}\right) + \frac{m-\sqrt{n}}{2}$$
$$= m + \sqrt{m^2 - n}.$$

It follows that $m = 4$ and $n = 5$. The requested sum is $10 \cdot 4 + 5 = 45$. □

Problem 1.5. The product

$$\left(\frac{1+1}{1^2+1} + \frac{1}{4}\right)\left(\frac{2+1}{2^2+1} + \frac{1}{4}\right)\left(\frac{3+1}{3^2+1} + \frac{1}{4}\right)\cdots\left(\frac{2022+1}{2022^2+1} + \frac{1}{4}\right)$$

can be written as $\frac{q}{2^r \cdot s}$, where r is a positive integer, and q and s are relatively prime odd positive integers. Find s.

Solution. Note that $\frac{n+1}{n^2+1} + \frac{1}{4} = \frac{1}{4} \cdot \frac{(n+2)^2+1}{n^2+1}$. Thus, the given product telescopes and is equal to

$$\frac{1}{4^{2022}} \cdot \frac{1}{1^2+1} \cdot \frac{1}{2^2+1} \cdot \left(2023^2+1\right)\left(2024^2+1\right) = \frac{\left(2023^2+1\right)\left(2024^2+1\right)}{2^{4045} \cdot 5}.$$

The last digit of 2023^2 must be 9 , so $2023^2 + 1$ is a multiple of 5 . It follows that the denominator of the reduced fraction is a power of 2 , and the value of s is 1 . □

Problem 1.6. Let a and b be positive real numbers satisfying

$$\frac{a}{b}\left(\frac{a}{b} + 2\right) + \frac{b}{a}\left(\frac{b}{a} + 2\right) = 2022.$$

Find the positive integer n such that

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \sqrt{n}$$

Solution. Adding 3 to both sides of the given equation yields

$$\left(\frac{a}{b}+\frac{b}{a}+1\right)^2=2025,$$

which implies that $\frac{a}{b}+\frac{b}{a}+1=45$. Then $\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^2=\frac{a}{b}+2+\frac{b}{a}=46$. In particular, the original equation is satisfied by $a=22+\sqrt{483}$ and $b=1$. □

Problem 1.7. Let a be a real number such that

$$5\sin^4\left(\frac{a}{2}\right)+12\cos a=5\cos^4\left(\frac{a}{2}\right)+12\sin a.$$

There are relatively prime positive integers m and n such that $\tan a=\frac{m}{n}$. Find $10m+n$.

Solution. Rewrite the given equation as

$$12\cos a-12\sin a=5\cos^4\left(\frac{a}{2}\right)-5\sin^4\left(\frac{a}{2}\right)$$

Then

$$12\cos a-12\sin a=5\left[\cos^2\left(\frac{a}{2}\right)-\sin^2\left(\frac{a}{2}\right)\right]=5\cos a$$

It follows that $\tan a=\frac{\sin a}{\cos a}=\frac{7}{12}$. The requested expression is $10\cdot 7+12=82$. □

Problem 1.8. The sum of the solutions to the equation

$$x^{\log_2 x}=\frac{64}{x}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Taking the logarithm base 2 of both sides of the equation gives $\log_2\left(x^{\log_2 x}\right)=\log_2\left(\frac{64}{x}\right)$, from which $(\log_2 x)(\log_2 x)=6-\log_2 x$. This is equivalent to $(\log_2 x-2)(\log_2 x+3)=0$, whose solutions are 4 and $\frac{1}{8}$. The sum of the solutions is $\frac{33}{8}$. The requested sum is $33+8=41$. □

Problem 1.9. Let α,β, and γ denote the angles of a triangle. Show that

$$\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2},$$
$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=4\sin\alpha\sin\beta\sin\gamma$$

and $\sin 4\alpha+\sin 4\beta+\sin 4\gamma=-4\sin 2\alpha\sin 2\beta\sin 2\gamma$.

Solution. Call the three identities (a), (b), and (c), respectively. If

$$\alpha+\beta+\gamma=\pi,$$

then

$$(\pi-2\alpha)+(\pi-2\beta)+(\pi-2\gamma)=\pi.$$

We can pass from (a) to (b), and also from (b) to (c), by substituting $\pi-2\alpha,\pi-2\beta$, and $\pi-2\gamma$ for α,β, and γ, respectively. It remains to verify (a), which can be done in many ways. For instance, substitute $2u,2v$, and $\pi-2u-2v$ for α,β, and γ, respectively. Then (a) becomes

$$\sin u\cos u+\sin v\cos v$$
$$=[2\cos u\cos v-\cos(u+v)]\sin(u+v).$$

Use the addition theorems of cosine and sine. □

Problem 1.10. Prove that no number in the sequence

$$11, 111, 1111, 11111, \ldots.$$

is the square of an integer.

Solution. If s is a number in the sequence, s must have the form

$$11 + 100m = 4(25m + 2) + 3,$$

where m is a non-negative integer, and therefore s leaves a remainder of 3 when divided by 4 . But squares are of the form $4n^2$ or $4n^2 + 4n + 1$ and hence leave remainders of either 0 or 1 when divided by 4 .

□

Problem 1.11. The length of the perimeter of a right triangle is 60 inches and the length of the altitude perpendicular to the hypotenuse is 12 inches. Find the sides of the triangle.

Solution. Let a, b, and c denote the sides, the last being the hypotenuse. The three parts of the condition are expressed by

$$\begin{aligned} a + b + c &= 60 \\ a^2 + b^2 &= c^2 \\ ab &= 12\text{c}. \end{aligned}$$

Observing that

$$(a + b)^2 = a^2 + b^2 + 2ab$$

we obtain

$$(60 - c)^2 = c^2 + 24c.$$

Hence $c = 25$ and either $a = 15, b = 20$ or $a = 20, b = 15$ (no difference for the triangle). □

Problem 1.12. Prove the proposition: If a side of a triangle is less than the average (arithmetic mean) of the two other sides, the opposite angle is less than the average of the two other angles.

Solution. Since $A + B + C = 180°$, proving that $A < (B + C)/2$ amounts to proving that $A < (180° - A)/2$ or $A < 60°$. But $A < 60°$ is equivalent to $\cos A > 1/2$, which suggests the use of the law of cosines.

By hypothesis, $b + c > 2a$. Squaring both sides and applying the law of cosines, we obtain

$$b^2 + 2bc + c^2 > 4\left(b^2 + c^2 - 2bc\cos A\right)$$

or

$$8bc\cos A > 3b^2 + 3c^2 - 2bc.$$

Subtracting $4bc$ from both sides, we obtain

$$4bc(2\cos A - 1) > 3(b - c)^2 \geqq 0.$$

Therefore $\cos A > 1/2$. □

Problem 1.13. Prove that the only solution of the equation

$$x^2 + y^2 + z^2 = 2xyz$$

in integers x, y, and z is $x = y = z = 0$.

Solution. Suppose x, y, and z are integers. Let 2^k, with $k \geqq 0$, be the highest power of 2 that divides x, y, and z, so that $x = 2^k x', y = 2^k y'$, and $z = 2^k z'$. Then substituting in the given equation and dividing through by 2^{2k}, we obtain

$$(x')^2 + (y')^2 + (z')^2 = 2^{k+1} x' y' z'.$$

Since the right-hand side is even, so is the left-hand side, and either x', y', and z' are all even or just one of them is. But if x', y', and z' are not all zero (and if one is, the others are), they cannot all be even, because 2 is not a common factor. Suppose x' is even and y' and z' are odd. Subtracting $(x')^2$ from both sides of the above equation yields

$$(y')^2 + (z')^2 = x' \left(2^{k+1} y' z' - x'\right).$$

Both $(y')^2$ and $(z')^2$ are of the form $4n^2 + 4n + 1$, and so the left-hand side divided by 4 leaves the remainder 2 , whereas the right-hand side is divisible by 4 (both x' and the quantity in parenthesis are even): A Contradiction. □

Problem 1.14. Solve the following system of three equations for the unknowns x, y and z :

$$\begin{aligned} 5732x + 2134y + 2134z &= 7866 \\ 2134x + 5732y + 2134z &= 670 \\ 2134x + 2134y + 5732z &= 11464 \end{aligned}$$

Solution. The simplest expression that is symmetric in x, y, and z is their sum. Adding the three proposed equations, we obtain

$$\begin{gathered} 10000x + 10000y + 10000z = 20000 \\ x + y + z = 2 \end{gathered}$$

By subtracting

$$2134x + 2134y + 2134z = 4268$$

from each of the three proposed equations, we obtain three new equations that when solved yield $x = 1, y = -1, z = 2$, respectively. □

Problem 1.15. A pyramid is called "regular" if its base is a regular polygon and the foot of its altitude is the center of its base. A regular pyramid has a hexagonal base the area of which is one quarter of the total surface-area S of the pyramid. The altitude of the pyramid is h. Express S in terms of h.

Solution. Pass a plane through the altitude of the pyramid and through the midpoint of one side (of length a) of its base. The intersection of this plane with the pyramid is an isosceles triangle that can be used as a key figure: its height is h, its legs are of length l (where l is the height of a lateral face of the pyramid), and its base is $2b$ (where b is the altitude of one of the six congruent equilateral triangles composing the base of the pyramid). The area of the base is

$$\frac{S}{4} = \frac{6ab}{2},$$

the area of the lateral surface is

$$\frac{3S}{4} = \frac{6al}{2},$$

and so

$$l = 3b.$$

Using the key figure, we obtain and so

$$h^2 + b^2 = l^2 = 9b^2$$

$$b^2 = \frac{h^2}{8}$$

We also have and so

$$b^2 + \frac{a^2}{4} = a^2$$

Therefore

$$a^2 = \frac{4b^2}{3} = \frac{h^2}{6}$$

$$S = 12ab = h^2\sqrt{3}.$$

□

Problem 1.16. Show that each number of the sequence

$$49, 4489, 444889, 44448889, \ldots$$

is a perfect square.(Recall the formula for the sum of a geometric progression)

Solution. Recall the formula for the sum of a geometric progression. The number of the sequence that has $2n$ digits is

$$\begin{aligned}
9 + 8(10 + &10^2 + \cdots + 10^{n-1}) \\
&+ 4\left(10^n + 10^{n+1} + \cdots + 10^{2n-1}\right) \\
&= 1 + (8 + 4 \cdot 10^n)\left(1 + 10 + \cdots + 10^{n-1}\right) \\
&= 1 + 4\left(10^n + 2\right)\left(10^n - 1\right)/(10 - 1) \\
&= \left(\frac{2 \cdot 10^n + 1}{3}\right)^2
\end{aligned}$$

This is the square of an integer, since

$$\left(2 \cdot 10^n + 1\right)/3 = 1 + 6\left(10^n - 1\right)/9 = 666 \cdots 67,$$

a number with n digits. □

Problem 1.17. Solve the system

$$2x^2 - 4xy + 3y^2 = 36$$
$$3x^2 - 4xy + 2y^2 = 36$$

(One solution is easy to guess, but you are required to find all solutions. Knowledge of analytic geometry is not needed to solve this problem, but may help to understand the result-how?)

Solution. We are required to find the points of intersection of two congruent ellipses symmetrical to each other with respect to the line $x=y$. Subtraction of the equations yields $x^2=y^2$. There are four points of intersection: $(6,6),(-6,-6),(2,-2),(-2,2)$. □

Problem 1.18. Solve the following system of three equations for the unknowns x, y, and z (a, b, and c are given) :

$$\begin{aligned} x^2y^2+x^2z^2 &= axyz, \\ y^2z^2+y^2x^2 &= bxyz, \\ z^2x^2+z^2y^2 &= cxyz. \end{aligned}$$

Solution. If $x=0$, the second (or third) equation yields $y^2z^2=0$, and so one more unknown, y or z, must also be 0 . Hence either x, y, and z are all different from 0 or at least two vanish. If any two vanish, the equations are satisfied.

Now we consider the case in which no one of the three unknowns is 0 . By dividing, we obtain the system

$$\begin{aligned} \frac{zx}{y}+\frac{xy}{z} &= a \\ \frac{yz}{x}+\frac{xy}{z} &= b, \\ \frac{yz}{x}+\frac{zx}{y} &= c \end{aligned}$$

Adding these three equations and dividing by 2, we have

$$\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}=\frac{a+b+c}{2}.$$

From this equation we subtract each of the three equations of the foregoing system and obtain

$$\begin{aligned} \frac{yz}{x} &= \frac{-a+b+c}{2} \\ \frac{zx}{y} &= \frac{a-b+c}{2} \\ \frac{xy}{z} &= \frac{a+b-c}{2}. \end{aligned}$$

The product of these three equations is

$$xyz=(-a+b+c)(a-b+c)(a+b-c)/8$$

which we divide by each equation of the foregoing system to obtain, after extracting a square root,

$$\begin{aligned} x &= [(a-b+c)(a+b-c)]^{1/2}/2 \\ y &= [(-a+b+c)(a+b-c)]^{1/2}/2 \\ z &= [(-a+b+c)(a-b+c)]^{1/2}/2. \end{aligned}$$

We must take into account, however, the two values of each square root. Let us concentrate upon a suggestive particular case and assume that a, b, and c are the lengths of the three sides of a triangle. Then by $(^\circ)$ above, xyz is positive, and therefore only the following four combinations of signs are admissible:

$$\begin{array}{ll} x & ++-- \\ y & +-+- \\ z & +--+ \end{array}$$

□

Problem 1.19. Prove: If n is an integer greater than $1, n^{n-1}-1$ is divisible by $(n-1)^2$.

Solution. We observe first that if n is greater than 1 , the quotient of $n^{n-1}-1$ and $n-1$ is

$$n^{n-2}+n^{n-3}+\cdots+n+1.$$

We conceive this sum as resulting from the polynomial

$$R(x)=(1+x)^{n-2}+(1+x)^{n-3}+\cdots+(1+x)+1$$

when we substitute in it $n-1$ for x. In the expansion of $R(x)$ in powers of x (you may use the binomial formula), the term independent of x is $R(0)=n-1$, and so

$$R(x)=Q_{n-3}(x)x+n-1$$

where $Q_{n-3}(x)$ is a polynomial of degree $n-3$ whose coefficients are integers. Now we substitute $n-1$ for x and collect our conclusions:

$$\begin{aligned} n^{n-1}-1 &= (n-1)R(n-1) \\ &= (n-1)\left[Q_{n-3}(n-1)(n-1)+n-1\right] \\ &= (n-1)^2\left[Q_{n-3}(n-1)+1\right]. \end{aligned}$$

□

Problem 1.20. Ten people are sitting around a round table. The sum of ten dollars is to be distributed among them according to the rule that each person receives one half of the sum that his two neighbors receive jointly. Is there just one way to distribute the money? Prove your answer.

Solution. Let $A, B, C, D, \ldots, J$ be the persons around the table, and $a, b, c, d, \ldots, j$ the amounts received by them, respectively; B is to the right of A, C to the right of $B, \ldots, A$ to the right of J. The rule is expressed by the equations $b=\frac{a+c}{2}, \quad c=\frac{b+d}{2}, \quad d=\frac{c+e}{2}, \ldots, \quad a=\frac{j+b}{2}$. First solution. From the above equations, it follows that

$$b-a=c-b=d-c=\ldots=a-j$$

so that everyone's share exceeds that of his neighbor on the left by the same amount. This constant excess must be zero, since

$$(b-a)+(c-b)+(d-c)+\ldots+(a-j)=0.$$

There is just one way to distribute the money: all shares are equal. Second solution. Some person (or persons) must receive the maximum amount. Let such a person be B. Then none of the numbers $a, \ldots, j$ is greater than b; and, in particular,

$$b-a \geqq 0, \quad b-c \geqq 0.$$

Yet, by the condition,

$$b-a=-(b-c).$$

Consequently, both of the two numbers $b-a$ and $b-c$ must be zero. Thus c also attains the maximum, as does d, and so on. Therefore $a=b=c=\ldots=j$. □